# A Derivation of the Long-Term Degradation of a Pulsed Atomic Frequency Standard from a Control-Loop Model*

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## ABSTRACT

The phase of a frequency standard that uses periodic interrogation and control of a local oscillator (LO) is degraded by a long-term random-walk component induced by downconversion of LO noise into the loop passband. The Dick formula for the noise level of this degradation is derived from an explicit solution of an LO control-loop model.

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# 1 INTRODUCTION

In 1987, following a suggestion of L. Cutler, G. J. Dick [1] described a source of long-term instability for a class of passive frequency standards, including those using ion traps and atomic fountains. Such a standard controls the frequency of a local oscillator (LO) by a feedback loop whose detection and control operations are periodic with period $T_c$. For each cycle, the output of the detector is a weighted average of the LO frequency error over the cycle. The weighting function $g(t)$ ($t$ = time), derived from quantum mechanical calculations, depends on the method by which the atoms are interrogated by the radio frequency field [1][2][3]. In general, $g(t)$ can be zero over a considerable portion of the cycle. The LO control signal is constant over a cycle, its level being some linear combination of the detector outputs from previous cycles.

The purpose of a frequency-control loop is to attenuate the frequency fluctuations of the LO inside the loop passband (long-term fluctuations), while tolerating them outside the passband (short-term fluctuations). As Dick saw, though, the periodic interrogation causes LO noise power, near the cycle frequency $f_c = 1/T_c$ and its harmonics, to be downconverted into the loop passband, thus injecting random false information about the current average LO frequency into the control signal. This random false frequency correction causes a component of white frequency modulation (FM), or random walk of phase, to persist in the output of the locked LO (LLO) over the long term. Dick gave a formula for the white FM noise level contributed by this effect, namely

$$S_y(0) = 2\sum_{k=1}^{\infty} \frac{g_k^2}{g_0^2} S_y^{\mathrm{LO}}(kf_c) \qquad (\;)$$

where $S_y(f)$ is the spectral density of the Dick effect portion of the fractional frequency of the LLO, $S_y^{LO}(f)$ is the spectral density of the fractional frequency of the free running LO, and $g_k$ is the Fourier cosine coefficient

$$g_k = \frac{1}{T_c}\int_0^{T_c} g(t)\cos(2\pi k f_c t)\,dt, \tag{2}$$

where $g(t)$ is assumed to be symmetric about $T_c/2$. Such a level of white FM near Fourier frequency zero contributes an asymptotic component of Allan variance given by

$$\sigma_y^2(\tau) \sim \frac{S_y(0)}{2\tau} \qquad (f_c\tau \to \infty).$$

Existing derivations [1][2][3] of the Dick formula (1) are partly intuitive, based on previous experience in the behavior of control loops. The intent of this study is to put the Dick effect on firmer ground by giving a mathematical derivation of (1) from a simple model for a periodic LO control loop with a general weighting function $g(t)$. On the way, an explicit solution for the output LLO frequency is derived. A careful interpretation of this solution yields a formula for the LLO spectral density $S_y(f)$, and conditions for the validity of the Dick formula.

## II. Control Loop Model

Figure 1 shows the chosen model for an LO control loop, containing both analog and digital elements. All signals are scaled as fractional frequency deviation from the ideal frequency determined by the atomic transition. The fractional frequency noise contributed by the free-running local oscillator is $y_{LO}(t)$. The output LLO fractional frequency deviation is $y(t)$. The result of interrogating $y(t)$ during the $n$th cycle of length $T_c$ is the error signal

$$\frac{1}{T_c g_0}\int_{(n-1)T_c}^{nT_c} g(t)y(t)\,dt, \tag{3}$$

where $g(t)$ is the sensitivity function, regarded as periodic with period $T_c$ for all real $t$, and

$$g_0 = \frac{1}{T_c}\int_0^{T_c} g(t)\,dt \neq 0.$$

In Figure 1, the interrogation of $y$ by $g$ in (3) is implemented by sampling the output of the linear time-invariant filter

$$G_?(\,) \quad \frac{1}{T_c g}\int_0^{T} g \quad )y( \quad du \tag{4}$$

at time $t = nT_c$. For later reference we note the transfer function of this filter:

$$G(f) = \frac{1}{T_c g_0}\int_0^{T_c} g(-t)e^{-i2\pi f t}\,dt. \tag{5}$$

The detection noise term $v_n$ can represent photon-count fluctuations in frequency standards with optical detection, for example. The cumulative sum of the error signals is $x_n$, which, multiplied by a constant $\lambda$, corrects the frequency of the LO for $nT_c < t \leq (n+1)T_c$. Except for initial conditions, the following two equations define the closed-loop model completely

$$x_n = x_n \quad \frac{1}{T_c g_0}\int_{(n-1)T_c}^{nT_c} g \quad )y, \quad + \quad v_n, \tag{6}$$

$$y(t) = y_{LO}(t) - \lambda x_n, \quad nT_c < t \leq (n \quad T_c \tag{7}$$

in which it is convenient to suppose that $n$ runs through all integers. This system has two inputs, $y_{LO}(t)$ and $v_n$, and one output, $y(t)$

## LLO FREQUENCY

It is evident from (7) that a solution for $x_n$ gives a solution for $y$ ) Substitution of (7) into (6) with $n$ replaced by $n$ gives the first order difference equation

$$\quad 1 \quad \lambda)x_n \quad w_n, \tag{8}$$

where

$$w_n \quad \frac{1}{T_c g_0} \int_{(n \quad 1)T_c}^{nT_c} g(t) y_{LO}(t) dt \quad v_n \tag{9}$$

If $0 < \lambda < 1$, then Eq (8) describes a lowpass filter. this case, the genera solution is

$$x_n = \sum_{i=0}^{\infty} (1 - \lambda)^i w_{n-j} \quad C(1 - \lambda)^n . \tag{.0}$$

From now on, we shall ignore the transient part of this solution by setting $C = 0$

Let us express $x_n$ directly as a function of the inputs $y_{LO}(t)$ and $v_n$. Define the discrete-time lowpass fil er $H_d$ with weights

$$h_n \quad \lambda(1 \quad \lambda)^n \quad n \geq 0,$$

whic sum to and ansfer function

$$H_d, f \quad \sum_{n=0}^{\infty} h_n e^{-i2\pi n f T_c} \qquad \frac{\lambda}{\lambda) e^{-i2\pi f}}$$

Substituting (9) into ( 0) gives

$$\begin{aligned} \lambda x_n & \quad \int_0^{\infty} h_c(t) y_{LO}(nT_c \quad) dt \quad H_d v_n \\ & = H_c y_{LO}(nT_c) + H_d v_n, \end{aligned} \tag{12}$$

where we have introduced a causal continuous time filter $H_c$. Its impulse response $h_c(t)$, defined piecewise for $t > 0$ by

$$h_c(t) = \frac{h_n}{T_c g_0} g(-t), \quad nT_c < t < (n+1)T_c, \quad n = 0, \ ,2, \ldots,$$

consists of of repetitions of a reversed cycle of $g$ with exponentially decreasing amplitudes. N tice a: $\int_0^{\infty} h_c(t) dt$ Its transfer function

$$H_c(f) \quad \int_0^{\infty} h_c( ) \quad ^{i2 \ ft} dt$$

satisfies

$$H_c(f) = H_d(f)G(f). \tag{13}$$

Substituting (12) into (7) gives a steady-state solution for the 1 ,1,() frequency:

$$y(t) = y_{\mathrm{LO}}(t) - H_c y_{\mathrm{LO}}(nT_c) + H_d v_n, \quad nT_c < t \leq (n+1)T_c \tag{14}$$

## IV. The LO Spectrum

Although (14) gives an explicit formula for the output frequency, its interpretation requires careful handling. Under reasonable assumptions (see below) on $y_{\mathrm{LO}}(t)$ and $v_n$ as random processes, we cannot expect the piecewise-defined process $y(t)$ to be stationary, or even to have stationary $n$th increments for some $n$. Thus, we do not know how to assign a spectral density to it. To get around this problem, it is convenient to study the the *samples* $x(nT_c)$ of the LLO time residual $x(t) = \int y(t)dt$. In turn, their behavior is determined by the behavior of the *average* LLO frequencies

$$\frac{x(nT_c) - x((n-1)T_c)}{T_c} = \frac{1}{T_c}\int_{(n-1)T_c}^{nT_c} y(t)dt = Ay(nT_c),$$

where $A$ is the moving-average filter whose action on a function $z(t)$ is

$$Az(t) = \frac{1}{T_c}\int_0^{T_c} z(t-u)du$$

Its transfer function is

$$A(f) = e^{-i\pi f T_c}\frac{\sin(\pi f T_c)}{\pi f T_c}. \tag{15}$$

If $z(t)$ is a constant $c$ for $(n-1)T_c < t < nT_c$, then $Az(nT_c) = c$. Therefore, applying $A$ to (14), with $n$ replaced by $n-1$, gives

$$Ay(nT_c) = Ay_{\mathrm{LO}}(nT_c) - H_c y_{\mathrm{LO}}((n-1)T_c) + H_d v_{n-1} \tag{16}$$

We are now going to derive the spectrum of the discrete-time process $\Delta y(nT_c)$ defined by (16). To this end, consider the auxiliary process defined by

$$Y(t) = Ay_{LO}(t) - H_c y_{LO}(t - T_c), \tag{17}$$

which is obtained from $y_{LO}(t)$ by a linear time-invariant operation $B$ with transfer function

$$B(f) = A(f) - e^{-i2\pi fT_c} H_c(f) = A(f) - e^{-i2\pi fT_c} H_d(f)G(f). \tag{18}$$

Assume that $y_{LO}(t)$ is a mean-continuous random process with stationary first increments [4] and a two sided (even) spectral density $S_y^{LO}(f)$, which necessarily satisfies

$$\int_0^{f_c} S_y^{LO}(f) f^2 df < \infty, \quad \int_{f_c}^{\infty} S_y^{LO}(f) df < \infty. \tag{19}$$

In particular, if $S_y^{LO}(f)$ is asymptotic to a power law $|f|^\alpha$ as $f \to 0$, then $\alpha > -3$. This class of noises allows all low-frequency power-law spectra customarily attributed to oscillators. Because $A(0) = H_d(0) = G(0) = 1$, we have

$$B(f) = O(f^2) \qquad (f \to 0); \tag{20}$$

hence $B$ attenuates any low-frequency divergence of $y_{LO}(t)$ allowed by (19), leaving a stationary process $Y(t)$ with two sided spectral density

$$S_Y(f) = |B(f)|^2 S_y^{LO}(f) \tag{21}$$

The first two terms of the right side of (16) are just $Y(t)$ sampled with period $T_c$. These samples, $Y(nT_c)$, constitute a discrete-time stationary process whose two sided spectral density is

$$\sum_{k=-\infty}^{\infty} S_Y(f + kf_c), \ |f| < f_c/2.$$

The terms with $k \neq 0$ account for the Dick effect. Let the detection noise process $v_n$ be independent of $y(t)$, $\phi(t)$ and stationary, with two sided spectral density $S_v(f)$. Then the process $\Delta y(nT_c)$ given by (16) is stationary. In view of the previous discussion, its two sided spectral density can be written as

$$S_{\Delta y}(f) = S^{0}_{\Delta y}(f) + S^{1}_{\Delta y}(f), \quad |f| \le f_c/2,$$

where

$$S^{0}_{\Delta y}(f) = S_Y(f) + |H_d(f)|^2 S_v(f), \tag{22}$$

the main part, so to speak, and

$$S^{1}_{\Delta y}(f) = \sum_{k \neq 0} S_Y(f + k f_c), \tag{23}$$

the aliased part, where the sum includes both positive and negative $k$.

## V. The Dick Formula

We are looking for a long-term white FM spectral component introduced by the aliased part. There is such a component if the aliased spectrum (23) is continuous at $f = 0$, and $S^{1}_{\Delta y}(0) > 0$. Sufficient conditions for the series in (23) to converge uniformly for $|f| \le f_c/2$ to a continuous function $S^{1}_{\Delta y}(f)$ are (i) $g(t)$ is square integrable on a $T_c$-period, and (ii) $S^{LO}_{y}(f)$ is continuous for $|f| > f_c/2$ and tends to zero as $f \to \infty$. To compute $S^{1}_{\Delta y}(0)$ we note from the transfer-function formulas (11) and (15) that

$$H_d(kf_c) = 1, \quad \Lambda(kf_c) = \delta_{k0},$$

where $\delta_{k0}$ is the Kronecker delta. Hence (18), (21), and (23) give

$$S^{1}_{\Delta y}(0) = 2\sum_{k=1}^{\infty} \left|G(kf_c)\right|^2 S^{LO}_{y}(kf_c), \tag{24}$$

where we have now used the symmetry of the summands about zero frequency. This formula holds for one-sided spectral densities also.

Observe that the numbers $|G(kf_c)|^2$ are invariant to translations of the function $g(t)$ in time. It follows that the result (24) is invariant to shifts in the time origin, i.e., if the LLO phase $x(t)$ is sampled on any time grid of form $nT_c + t_0$, then the samples will include a white-FM component with spectral density (24) at zero frequency. Moreover, if the time origin can be chosen so that the periodic function $g(t)$ is even, then it is also even about $T_c/2$, and

$$G(kf_c) = \frac{g_k}{g_0},$$

where $g_k$ is given by (2). Thus (24) reduces to the Dick formula (1).

## VI. Remarks

If the actual LLO frequency were $Y(t)$ instead of $y(t)$, there would be no Dick effect. Unfortunately, $Y(t)$ is only a tool for the analysis; its existence is mathematical, not physical.

The Dick effect may be hidden by the main part (22) of the LLO spectrum. If the detection noise $v_n$ is white, then the term $|H_d(f)|^2 S_v(f)$ competes directly with the Dick effect as another white-FM noise at low frequencies. The basic action of the control loop operates on the LO frequency by a filter with transfer function $B(f) = A(f) - e^{-i2\pi f T_c} H_c(f)$, which, as we observed, is $O(f^2)$ as $f \to 0$. Thus, the filter adds 2 to the exponent of any low-frequency power law that $S_y^{LO}(f)$ obeys. If $S_y^{LO}(f)$ is more divergent than $f^{-2}$ (random walk FM), then $S^0_{Ay}(f)$ is unbounded near $f = 0$, hence masks the Dick effect completely. Random walk FM in the LO is transformed to another white FM component in the LLO.

Anything less divergent, like $f^{-1}$(flicker FM), is transformed to an 1,1,() spectral density that tends to zero at low frequencies. In this case, the 1 )ick effect and the detection noise predominate in the long term.

Although this derivation is confined to a particular loop model, the precision with which the 1 )ick formula emerges leads the author to conjecture that, for a given sensitivity function $g(t)$, the formula 1101(1s for any feedback mechanism that serves the fundamental pur ])os(! of steering the 1 ,() to the atomic-line frequency.

## REFERENCES


[1] G. J. Dick, "Local oscillator induced instabilities in trapped ion frequency standards", *Proc 19th Precise Time and Time Interval (PTTI) Applications and Planning Meeting*, pp 133 147, Redondo Beach, CA, 1987

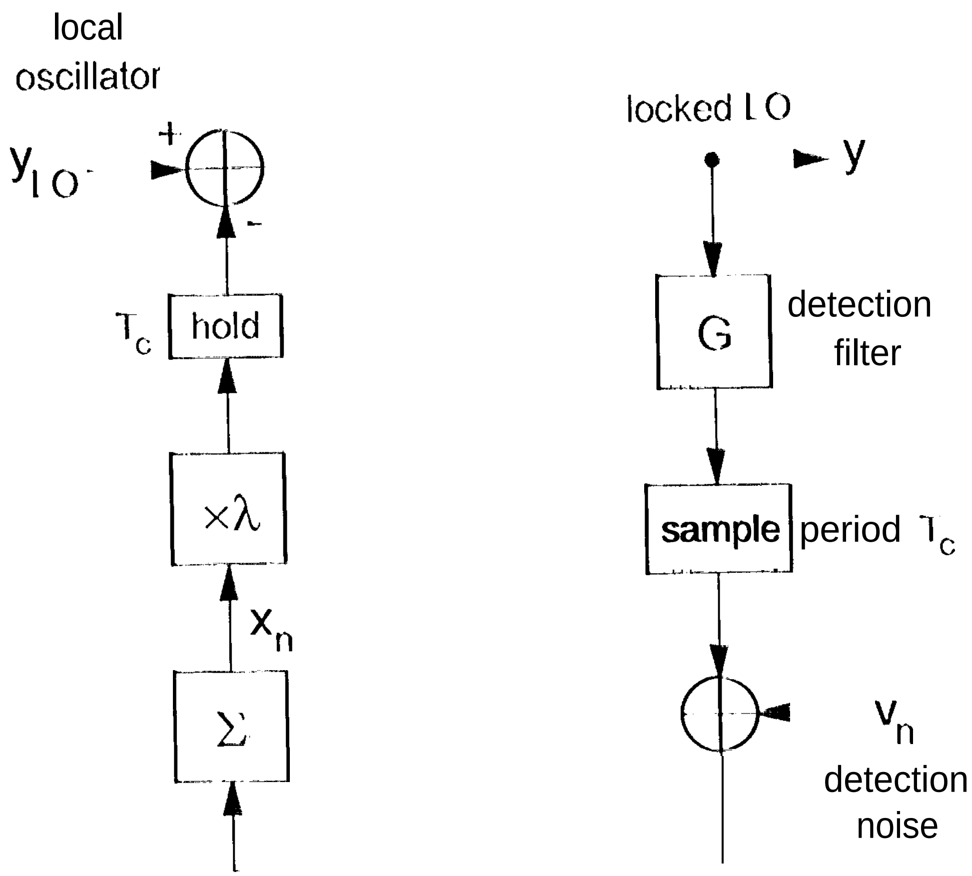
[2] G. J. Dick, J. D. Prestage, C. A. Greenhall, L. Maleki, "Local oscillator induced degradation of medium-term stability in passive atomic frequency standards", *Proc 22nd PTTI Meeting*, pp 487 508, Vienna, VA, 1990

[3] G. Santarelli, Ph. Laurent, A. Clairon, G. J. Dick, C. A. Greenhall, C. Audoin, "Theoretical description and experimental evaluation of the effect of the interrogation oscillator frequency noise on the stability of a pulsed atomic frequency standard", *10th European Frequency and Time Forum*, Brighton, UK, 1996

[4] A. M. Yaglom, *An Introduction to the Theory of Stationary Random Functions*. New Jersey: Prentice-Hall, 1962, pp 86 93

## FIGURE CAPTION

Fig. 1. A feedback-loop model for a local oscillator with periodic interrogation and control. The impulse response 0: the filter $G$ is one cycle of the normalized, reversed interrogation sensitivity function ~(f).



local
oscillator
$y_{LO}$
+
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